以伯恩斯坦多項式求解線性時變廣義系統的數值解

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摘要

本論文是以伯恩斯坦多項式的數值轉換為重點，再以此方法應用於求解線性時變廣義系統方程式的數值解。論文集中在建構一個高階的運算矩陣，在連續信號的轉換上，能獲得最佳的效能。我們不僅分析基底所形成的運算矩陣，也求解它們的積分運算矩陣、乘積矩陣和係數矩陣。主要的特性技巧是轉換一個廣義系統方程式成一個代數方程式。因此，求解的過程可降低或簡化。伯恩斯坦多項式的優點是比片斷常數的正交函數更能夠產生較精確的數值解。我們提出了可利用的最佳值，能夠最小化數值解的相對誤差。而且，透過已知的解析解計算誤差估計，以及討論在廣義系統方程式運算上的效能。

關鍵字：線性時變；廣義系統；伯恩斯坦多項式；積分運算矩陣；乘積矩陣；係數矩陣

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Numerical solutions of linear time-varying descriptor systems via Bernstein polynomials

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Abstract

State analysis of linear time-varying descriptor systems via Bernstein polynomials are proposed in this paper. Based upon some useful properties of Bernstein polynomials, an operational matrix of integration, a special product matrix and a related coefficient matrix are applied to solve the descriptor systems. The unknown coefficient matrix will be in generalized Lyapunov equation form, which is solved via the Kronecker product method. The advantages of Bernstein polynomials are not only the values of m are adjustable but also it can yield more accurate numerical solutions than orthogonal functions to the problems of solving the descriptor systems. We propose that the available optimal values of m can minimize the relative errors in the numerical solutions. The high accuracy and the wide applicability of Bernstein polynomials approach will be demonstrated with numerical examples.

Keywords：linear time-varying; descriptor systems; Bernstein polynomials; operational matrix of integration; product matrix; coefficient matrix

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I. Introduction

The descriptor equation (DAE) is applied in many fields such as physical models, electrical networks, singular perturbation, control theory problems, constrained mechanical systems of rigid bodies, and discretization of partial differential equations, etc. (Brenan, Campbell, & Petzold (1989), Campbell (1982), Griepentrog (1989)). Some numerical methods via backward differentiation formula (BDF) (Brenan et al. (1989), Ascher (1989), Ascher & Petzold (1991), Babolian & Hosseini (2003), Gear & Petzold (1984)), differential transform (Ayaz (2004), Ayaz (2004)) and implicit Runge-Kutta (IRK) methods (Brenan et al. (1989), Ascher et al. (1991), Ascher & Spiter(1994)) have been developed to solve the DAEs.

Padé series approximation (Celik & Bayram (2003), Celik & Bayram (2003)) was considered to solve the DAEs, while Ayaz's method (Ayaz (2004)) is much easier. Ayaz's method can yield reliable results through simple operation and also overcome many obstacles to some other numerical methods. Nevertheless, Liu and Song (Liu & Song (2007)) presented a counter example to account for the fact that Ayaz's method is not suitable for all index-3 DAEs.

Differential transform method was first introduced by Zhou (Zhou (1986)) to solve linear and nonlinear electrical circuit problems. Chen and Ho (Chen & Ho (1999)) followed to apply the method to solve partial differential equations. Ayaz (Ayaz (2004)) used it on the system of differential equations as well. The related results were in (Ayaz (2004), Chen et al. (1999), Abdel-Halim Hassan (2002)). Differential transform method has the advantage over some numerical methods such as BDF and IRK in reducing the difficulties for index-2 DAEs (Ayaz (2004), Ayaz (2004)). Some numerical methods (Ascher (1989), Ascher & Petzold (1991), Babolian & Hosseini (2003), Gear & Petzold (1984)) developed for the solution of DAEs are only suitable for low index DAEs or high index DAEs with special structure (Higueras & Garcýa-Celayeta (1999)). Bernstein polynomials have been applied recently to solve some linear and nonlinear differential equations by Bhatta and Bhatti (Bhatta & Bhatti (2006)), Bhatti and Bracken (Bhatti & Bracken (2007)). The useful mathematical tool Bernstein polynomials are extensively used on optimal control (Behroozifara, M. et al. (2014)), signal processing in communications (Caglar & Akansu (1993), Baradarani, A. et al. (2010), Kumar, A. et al. (2014), Suman, S. et al. (2015)) and mechanics research (Ernesto, Charlie A. (2010)).

In this paper, we present the properties of Bernstein polynomials and apply the Bernstein transform with optimal order to solve the linear time-varying descriptor systems. Many scholars just mentioned that Bernstein polynomials could be utilized to solve the differential equations and systems. They neglected a question−how large the respective ranks m should be to obtain more accurate numerical solutions. We propose that the available optimal values of m can minimize the relative errors in the numerical solutions.

II. Some properties of Bernstein polynomials

A. Polynomial basis

The B-polynomial of n-th degree are defined on the interval $[0,1]$ as (Bhatti et al. (2007)).

$$B_{i,n}(t) = \binom{n}{i} t^i \frac{(R-t)^{n-i}}{R^n}, 0 \le i \le n, \tag{1}$$

for $i = 0,1,\dots,n$, where the binomial coefficients are the combinations given by

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, \tag{2}$$

and R is the maximum such that the polynomials are defined to form a complete basis over the interval $[0,R]$. These polynomials are written down handily. As i increases by 1, the exponent on the t-term increases by 1 while the exponent on the $(R-t)$-term decreases by 1. For example, a set of 8 polynomials of degree 7:

$$\begin{aligned}
B_{0,7}(t) &= (1-t)^7,\\
B_{1,7}(t) &= 7t(1-t)^6,\\
B_{2,7}(t) &= 21t^2(1-t)^5,\\
B_{3,7}(t) &= 35t^3(1-t)^4,\\
B_{4,7}(t) &= 35t^4(1-t)^3,\\
B_{5,7}(t) &= 21t^5(1-t)^2,\\
B_{6,7}(t) &= 7t^6(1-t),\\
B_{7,7}(t) &= t^7.
\end{aligned}$$

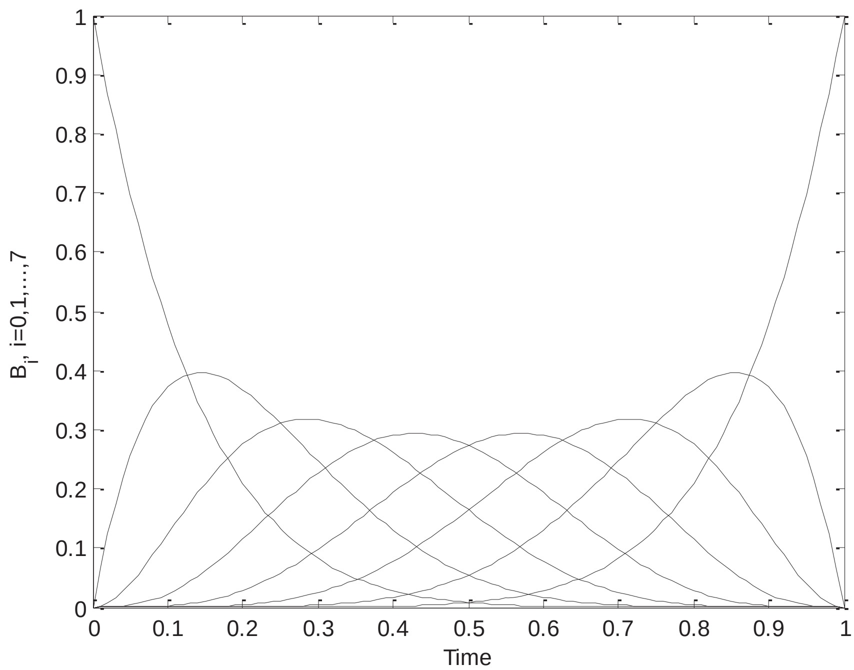

Figure 1. The set of 8 B-polynomials of degree 7 are shown in the region x=[0,1].

We illustrate it in Figure 1. The recursive definition of the B-polynomials over this interval is generated below:

$$B_{i,n}(t)=\frac{(R-t)}{R}B_{i,n-1}(t)+\frac{t}{R}B_{i-1,n-1}(t). \tag{3}$$

The derivatives of the n-th degree B-polynomials are given by

$$\frac{dB_{i,n}(t)}{dt}=\frac{n}{R}(B_{i-1,n-1}(t)-B_{i,n-1}(t)). \tag{4}$$

We see that each of the B-polynomials is positive and the sum of all the B-polynomials is 1, or $\sum_{i=0}^{n}B_{i,n}(t)=1$, on the interval $[0,R]$. It is shown that any given polynomial of degree n can be expanded in terms of the B-polynomials:

$$P(t)=\sum_{i=0}^{n}C_i\,B_{i,n}(t),\quad n\geq 1. \tag{5}$$

Any set of functions having these properties is called a partition of unity on the interval $[0,R]$. B-splines have similar properties and have been extensively used in most programs for computer-aided design. It is perhaps instructive to look at the graphs of the individual B-polynomials of the sums and to visualize how they might add to approximate a function $y(t)$.

B. Function approximation

Any function $y(t)$ which is square integrable in the interval $[0,1]$ can be expanded in Bernstein polynomials with an infinite number of terms

$$y(t) = \sum_{i=0}^{\infty} c_i \, b_i(t), \; t \in [0,1]. \tag{6}$$

Usually, the polynomial expansion (6) contains an infinite number of terms for a smooth $y(t)$. If $y(t)$ is a continuous function, then the sum in (6) may be terminated after m terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i \, b_i(t) = \boldsymbol{c}_{(m)}^T \boldsymbol{b}_{(m)}(t) \stackrel{\Delta}{=} y^*(t), \; t \in [0,1], \tag{7}$$

$$\boldsymbol{c}_{(m)} \stackrel{\Delta}{=} [\, c_0 \;\; c_1 \;\; \cdots \;\; c_{m-1}]^T, \tag{8}$$

$$\boldsymbol{b}_{(m)}(t) \stackrel{\Delta}{=} [\, b_0(t) \;\; b_1(t) \;\; \cdots \;\; b_{m-1}(t)]^T, \tag{9}$$

where "T" indicates transposition, the subscript m in the parentheses denotes their dimensions, $y^*(t)$ denotes the truncated sum. Let us define the m-square Bernstein matrix as

$$B_{(m\times m)} \stackrel{\Delta}{=} [\, \boldsymbol{b}_{(m)}(\frac{1}{2m}) \;\; \boldsymbol{b}_{(m)}(\frac{3}{2m}) \;\; \cdots \;\; \boldsymbol{b}_{(m)}(\frac{2m-1}{2m})]. \tag{10}$$

Substituting $t = \frac{1}{2m}, \frac{3}{2m}, \cdots, \frac{2m-1}{2m}$ into (6) yields

$$\begin{aligned} y_{(m)}^* &\stackrel{\Delta}{=} [\, y^*(\frac{1}{2m}) \;\; y^*(\frac{3}{2m}) \;\; \cdots \;\; y^*(\frac{2m-1}{2m})] \\ &= \boldsymbol{c}_{(m)}^T B_{(m\times m)}. \end{aligned} \tag{11}$$

It is obvious that

$$\boldsymbol{c}_{(m)}^T = y_{(m)}^* B_{(m\times m)}^{-1}. \tag{12}$$

Equation (12) is called the forward transform, which transforms the time function $y_{(m)}^*$ into the coefficient vector $\boldsymbol{c}_{(m)}^T$, and (11) is called the inverse transform, which recovers $y_{(m)}^*$ from $\boldsymbol{c}_{(m)}^T$.

In practical applications, a small number of terms will increase the speed of the calculation and save memory storage, while a large number of terms will improve the resolution. Therefore, a trade-off between the calculation speed, memory saving and the resolution should be taken in the analysis.

C. Integration of Bernstein polynomials

In Bernstein polynomials analysis for a dynamic system, all functions need to be transformed into Bernstein polynomials. The integration of Bernstein polynomials can be expanded into Bernstein polynomials with Bernstein coefficient matrix P.

$$\int_0^t \boldsymbol{b}_{(m)}(\tau)d\tau \approx P_{(m\times m)}\boldsymbol{b}_{(m)}(t),\ t \in [0,1], \tag{13}$$

where the m-square matrix P defined below is called the *operational matrix* of the Bernstein integral. Consider $m = 8$

$$P_{(8\times 8)} = \begin{bmatrix} 1.1735e-006 & 1.4275e-001 & \cdots & 1.2500e-001 \\ -8.2143e-006 & 7.2421e-004 & \cdots & 1.2499e-001 \\ \vdots & \vdots & \ddots & \vdots \\ -1.1735e-006 & 1.0346e-004 & \cdots & 1.2500e-001 \end{bmatrix}. \tag{14}$$

D. Multiplication of Bernstein polynomials

In the study of time-varying system via Bernstein polynomials, it is usually necessary to evaluate $\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^T(t)$. Let $\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^T(t) =^{\Delta} M_{(m\times m)}(t)$ which is called the *product matrix* of Bernstein polynomials. That is,

$$M_{(m\times m)}(t) \triangleq \begin{bmatrix} b_0b_0 & b_0b_1 & b_0b_2 & b_0b_3 & \cdots & b_0b_{m-1} \\ b_1b_0 & b_1b_1 & b_1b_2 & b_1b_3 & \cdots & b_1b_{m-1} \\ b_2b_0 & b_2b_1 & b_2b_2 & b_2b_3 & \cdots & b_2b_{m-1} \\ b_3b_0 & b_3b_1 & b_3b_2 & b_3b_3 & \cdots & b_3b_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m-1}b_0 & b_{m-1}b_1 & b_{m-1}b_2 & b_{m-1}b_3 & \cdots & b_{m-1}b_{m-1} \end{bmatrix}. \tag{15}$$

The matrix $M_{(m\times m)}(t)$ satisfies

$$M_{(m\times m)}(t)\boldsymbol{c}_{(m)} = C_{(m\times m)}\boldsymbol{b}_{(m)}(t), \tag{16}$$

where $\boldsymbol{c}_{(m)}$ is defined as (8). Consider $m = 8$, the *coefficient matrix* $C_{(8\times 8)}$ is defined as follows:

$$C_{(8\times 8)} = \begin{bmatrix} c_0 & c_1 - c_0 & \cdots & 6468c_1 - 1716c_0 - 9702c_2 + 7350c_3 - 2940c_4 + 588c_5 - 49c_6 + c_7 \\ 0 & c_0 & \cdots & 6468c_0 - 22638c_1 + 30870c_2 - 20580c_3 + 6860c_4 - 1029c_5 + 49c_6 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 \end{bmatrix}, \tag{17}$$

$$\boldsymbol{c}_{(8)} \triangleq [\, c_0, c_1, c_2, \dots, c_7]^T, \tag{18}$$

and

$$\boldsymbol{b}_{(8)}(t) \triangleq [\, b_0(t), b_1(t), b_2(t), \dots, b_7(t)]^T, \tag{19}$$

where

$$\left.\begin{aligned} b_0(t) &= -t^7 + 7t^6 - 21t^5 + 35t^4 - 35t^3 + 21t^2 - 7t + 1 \\ b_1(t) &= 7t^7 - 42t^6 + 105t^5 - 140t^4 + 105t^3 - 42t^2 + 7t \\ b_2(t) &= -21t^7 + 105t^6 - 210t^5 + 210t^4 - 105t^3 + 21t^2 \\ b_3(t) &= 35t^7 - 140t^6 + 210t^5 - 140t^4 + 35t^3 \\ b_4(t) &= -35t^7 + 105t^6 - 105t^5 + 35t^4 \\ b_5(t) &= 21t^7 - 42t^6 + 21t^5 \\ b_6(t) &= 7t^6 - 7t^7 \\ b_7(t) &= t^7 \end{aligned}\right\}, \ 0 \le t \le 1. \tag{20}$$

With the powerful properties of (13), (15) and (16), the state solution of linear time-varying descriptor systems can be easily found.

III. State analysis of linear time-varying descriptor systems via Bernstein polynomials

Consider a general linear time-varying descriptor system with the form:

$$E(t)\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{21}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^{\tilde{n}}$, the control variable $\boldsymbol{u}(t) \in \boldsymbol{R}^r$, $E(t) \in \boldsymbol{R}^{\tilde{n}\times\tilde{n}}$, $A(t) \in \boldsymbol{R}^{\tilde{n}\times\tilde{n}}$ and $B(t) \in \boldsymbol{R}^{\tilde{n}\times r}$. The response $\boldsymbol{x}(t)$, $0 \le t \le t_f$, is required to be found. Since the hybrid function is defined for $[0,1]$, the time scale should be normalized by defining $t = t_f \cdot \tau$. Then (21) becomes

$$\begin{aligned} E(\tau)\dot{\boldsymbol{x}}(\tau) &\overset{\Delta}{=} E(\tau)\frac{d\boldsymbol{x}(\tau)}{d\tau} \\ &= t_f[A(\tau)\boldsymbol{x}(\tau) + B(\tau)\boldsymbol{u}(\tau)], \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{aligned} \tag{22}$$

where $\tau \in [0,1]$. Let $t_f \cdot A(\tau)$ and $t_f \cdot B(\tau)\boldsymbol{u}(\tau)$ be decomposed into the following form:

$$\begin{aligned} t_f \cdot A(\tau) &= A_1\,\alpha_1(\tau) + A_2\,\alpha_2(\tau) + \cdots + A_p\,\alpha_p(\tau) \\ &= \sum_{i=1}^{p} A_i\ \alpha_i(\tau), \ \ (p \le \tilde{n}^2), \end{aligned} \tag{23}$$

where each A_i is a $\tilde{n} \times \tilde{n}$ constant matrix, and $\alpha_i(\tau)$ is a scalar time function;

$$\begin{aligned} t_f \cdot B(\tau)\boldsymbol{u}(\tau) &= b_1\,\beta_1(\tau) + b_2\,\beta_2(\tau) + \cdots + b_q\,\beta_q(\tau) \\ &= \sum_{j=1}^{q} b_j\ \beta_j(\tau), \ \ (q \le \tilde{n}), \end{aligned} \tag{24}$$

where each b_j is a constant $\tilde{n}$-vector, and $\beta_j(\tau)$ is a scalar time function.

$$\begin{aligned} E(\tau) &= E_1\,\gamma_1(\tau) + E_2\,\gamma_2(\tau) + \cdots + E_r\,\gamma_r(\tau) \\ &= \sum_{k=1}^{r} E_k\ \gamma_k(\tau),\ \ (r \le \tilde{n}^2), \end{aligned} \tag{25}$$

where each E_k is a $\tilde{n} \times \tilde{n}$ constant matrix, and $\gamma_k(\tau)$ is a scalar time function. Thus, (22) becomes

$$\begin{aligned} &\sum_{k=1}^{r} E_k\ \gamma_k(\tau)\dot{\boldsymbol{x}}(\tau) \\ &= \sum_{i=1}^{p} A_i\ \alpha_i(\tau)\boldsymbol{x}(\tau) + \sum_{j=1}^{q} b_j\ \beta_j(\tau), \quad \boldsymbol{x}(0) \\ &= \boldsymbol{x}_0. \end{aligned} \tag{26}$$

These $\alpha_i(\tau)$ and $\beta_j(\tau)$ can be expanded in the Bernstein polynomials over the time interval $[0,1]$.

$$\begin{aligned} \alpha_i(\tau) &= d_{i0}\,b_0(\tau) + d_{i1}\,b_1(\tau) + \cdots + d_{i,m-1}\,b_{m-1}(\tau) \\ &= \boldsymbol{d}_i^T\,\boldsymbol{b}_{(m)}(\tau), \end{aligned} \tag{27}$$

$$\begin{aligned} \beta_j(\tau) &= e_{j0}\,b_0(\tau) + e_{j1}\,b_1(\tau) + \cdots + e_{j,m-1}\,b_{m-1}(\tau) \\ &= \boldsymbol{e}_j^T\,\boldsymbol{b}_{(m)}(\tau), \end{aligned} \tag{28}$$

$$\begin{aligned} \gamma_k(\tau) &= f_{k0}\,b_0(\tau) + f_{k1}\,b_1(\tau) + \cdots + f_{k,m-1}\,b_{m-1}(\tau) \\ &= \boldsymbol{f}_k^T\,\boldsymbol{b}_{(m)}(\tau), \end{aligned} \tag{29}$$

where

$$\boldsymbol{d}_i = [d_{i0}\ \ d_{i1}\ \ \cdots\ \ d_{i,m-1}]^T,\ \ i = 1,2,\cdots,\tilde{n}^2,$$

$$\boldsymbol{e}_j = [e_{j0}\ \ e_{j1}\ \ \cdots\ \ e_{j,m-1}]^T,\ \ j = 1,2,\cdots,\tilde{n},$$

$$\boldsymbol{f}_i = [f_{k0}\ \ f_{k1}\ \ \cdots\ \ f_{k,m-1}]^T,\ \ k = 1,2,\cdots,\tilde{n}^2.$$

Likewise, $\dot{\boldsymbol{x}}(\tau)$ is expanded into the Bernstein polynomials:

$$\dot{\boldsymbol{x}}(\tau) = \begin{bmatrix} g_{10} & g_{11} & \cdots & g_{1,m-1} \\ g_{20} & g_{21} & \cdots & g_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ g_{\tilde{n}0} & g_{\tilde{n}1} & \cdots & g_{\tilde{n},m-1} \end{bmatrix} \begin{bmatrix} b_0(\tau) \\ b_1(\tau) \\ \vdots \\ b_{m-1}(\tau) \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{g}_1^T \\ \boldsymbol{g}_2^T \\ \vdots \\ \boldsymbol{g}_{\tilde{n}}^T \end{bmatrix} \boldsymbol{b}_{(m)}(\tau) \triangleq G\, \boldsymbol{b}_{(m)}(\tau), \tag{30}$$

where G is an $\tilde{n} \times m$ matrix and $\boldsymbol{g}_i = [g_{i0} \;\; g_{i1} \;\; \cdots \;\; g_{i,m-1}]^T$. Integration of (30) yields

$$\begin{aligned} \boldsymbol{x}(\tau) &= \int_0^\tau \dot{\boldsymbol{x}}\,(\hat{\tau})d\hat{\tau} + \boldsymbol{x}_0 = G\int_0^\tau \boldsymbol{b}_{(m)}\,(\hat{\tau})d\hat{\tau} + \boldsymbol{x}_0 \\ &= \{G\,P + [\underbrace{\boldsymbol{x}_0 \;\; \cdots \;\; \boldsymbol{x}_0}_{(m)\ \text{columns}}]B^{-1}\}\boldsymbol{b}_{(m)}(\tau). \end{aligned} \tag{31}$$

Substituting (30) and (31) into (26), we obtain

$$\begin{aligned} &\sum_{k=1}^{r} E_k\ \gamma_k(\tau) G \boldsymbol{b}_{(m)}(\tau) \\ &= \sum_{i=1}^{p} A_i\ \alpha_i(\tau)\{G\,P \\ &+ [\underbrace{\boldsymbol{x}_0 \;\; \cdots \;\; \boldsymbol{x}_0}_{(m)\ \text{columns}}]B^{-1}\}\boldsymbol{b}_{(m)}(\tau) + \sum_{j=1}^{q} b_j\ \beta_j(\tau). \end{aligned} \tag{32}$$

By (27), (28) and (29), the state equation (32) becomes

$$\begin{aligned} &\sum_{k=1}^{r} E_k\,[\boldsymbol{f}_k^T\ \boldsymbol{b}_{(m)}(\tau)] G \boldsymbol{b}_{(m)}(\tau) \\ &= \sum_{i=1}^{p} A_i\,[\boldsymbol{d}_i^T\ \boldsymbol{b}_{(m)}(\tau)]\{G\,P + [\underbrace{\boldsymbol{x}_0 \;\; \cdots \;\; \boldsymbol{x}_0}_{(m)\ \text{columns}}]B^{-1}\}\boldsymbol{b}_{(m)}(\tau) \\ &+ \sum_{j=1}^{q} b_j\,[\boldsymbol{e}_j^T\ \boldsymbol{b}_{(m)}(\tau)]. \end{aligned} \tag{33}$$

Applying (16) to (33), we rewrite (33) as:

$$\sum_{k=1}^{r} E_k \; G \, F_k \boldsymbol{b}_{(m)}(\tau) = \sum_{i=1}^{p} A_i \{G \, P + [\underbrace{\boldsymbol{x}_0 \; \cdots \; \boldsymbol{x}_0}_{(m)\ \text{columns}}] B^{-1}\} D_i \boldsymbol{b}_{(m)}(\tau) + \sum_{j=1}^{q} b_j \, \boldsymbol{e}_j^T \, \boldsymbol{b}_{(m)}(\tau), \tag{34}$$

where $F_k \boldsymbol{b}_{(m)}(\tau) = M_{(m\times m)}(\tau) \boldsymbol{f}_k = \boldsymbol{b}_{(m)}(\tau) \boldsymbol{b}_{(m)}^T(\tau) \boldsymbol{f}_k$ and $D_i \boldsymbol{b}_{(m)}(\tau) = M_{(m\times m)}(\tau) \boldsymbol{d}_i = \boldsymbol{b}_{(m)}(\tau) \boldsymbol{b}_{(m)}^T(\tau) \boldsymbol{d}_i$ is a copy of (16). Therefore, equation (34) becomes

$$\sum_{k=1}^{r} E_k \; G \, F_k = \sum_{i=1}^{p} A_i \{G \, P + [\boldsymbol{x}_0 \; \cdots \; \boldsymbol{x}_0] B^{-1}\} D_i + \sum_{j=1}^{q} b_j \, \boldsymbol{e}_j^T, \tag{35}$$

where the dimensional subscripts have been dropped to simplify the notation. The components of G can be obtained by solving (35). To facilitate the solution, let

$$K \triangleq \sum_{i=1}^{p} A_i \, [\boldsymbol{x}_0 \; \cdots \; \boldsymbol{x}_0] B^{-1} D_i + \sum_{j=1}^{q} b_j \, \boldsymbol{e}_j^T, \tag{36}$$

and then

$$K = \begin{bmatrix} k_{10} & k_{11} & \cdots & k_{1,m-1} \\ k_{20} & k_{21} & \cdots & k_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ k_{\tilde{n}0} & k_{\tilde{n}1} & \cdots & k_{\tilde{n},m-1} \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{k}_1^T \\ \boldsymbol{k}_2^T \\ \vdots \\ \boldsymbol{k}_{\tilde{n}}^T \end{bmatrix}, \; \boldsymbol{k}_j = [k_{j0} \;\; k_{j1} \;\; \cdots \;\; k_{j,m-1}]^T. \tag{37}$$

Equation (35) is rewritten as

$$\sum_{k=1}^{r} E_k \; G \, F_k = \sum_{i=1}^{p} A_i \; G \, P \, D_i + K. \tag{38}$$

The unknown matrix G in the above equation is premultiplied by E_k and A_i and postmultiplied by F_k and $P \cdot D_i$. This is called the generalised Lyapunov equation, which is solved by using the Kronecker product $\otimes$. Manipulation of (38) gives

$$\begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_{\tilde{n}} \end{bmatrix} = \{\sum_{k=1}^{r} E_k \otimes F_k^T - \sum_{i=1}^{p} A_i \otimes (P \cdot D_i)^T\}^{-1} \begin{bmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \\ \vdots \\ \boldsymbol{k}_{\tilde{n}} \end{bmatrix}. \tag{39}$$

Once $\boldsymbol{g}_i$, $i = 1,2,\cdots,\tilde{n}$, are known, the state variable $\boldsymbol{x}(\tau)$ in the Bernstein polynomials expansion can be calculated from (31).

IV. Numerical examples

Example 1. Consider a index-2 descriptor system (Ayaz (2004), Celik et al. (2003)) as follows:

$$\begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1+t \\ 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{40}$$

If we solve (40) for $\boldsymbol{x}(t)$ directly, the analytic solution for $\boldsymbol{x}(t)$ can be shown to be $\boldsymbol{x}(t) = [\mathrm{e}^{-t} + t\sin(t) \;\; \sin(t)]^T$. The comparison between the Bernstein polynomials solution $\tilde{\boldsymbol{x}}(t)$ and the analytic solution $\boldsymbol{x}(t)$ for $t \in [0,1]$ is shown in Figure 2 for $m = 11$, which confirms that the Bernstein polynomials approach gives almost the same solution of the analytic method. The average relative errors of our method and Ayaz's method (Ayaz (2004)) are $[3.816009912165994 \cdot 10^{-9} \;\; 5.529388863882184 \cdot 10^{-11}]$ and $[5.647333072151591 \cdot 10^{-9} \;\; 1.271038155751912 \cdot 10^{-9}]$, respectively.

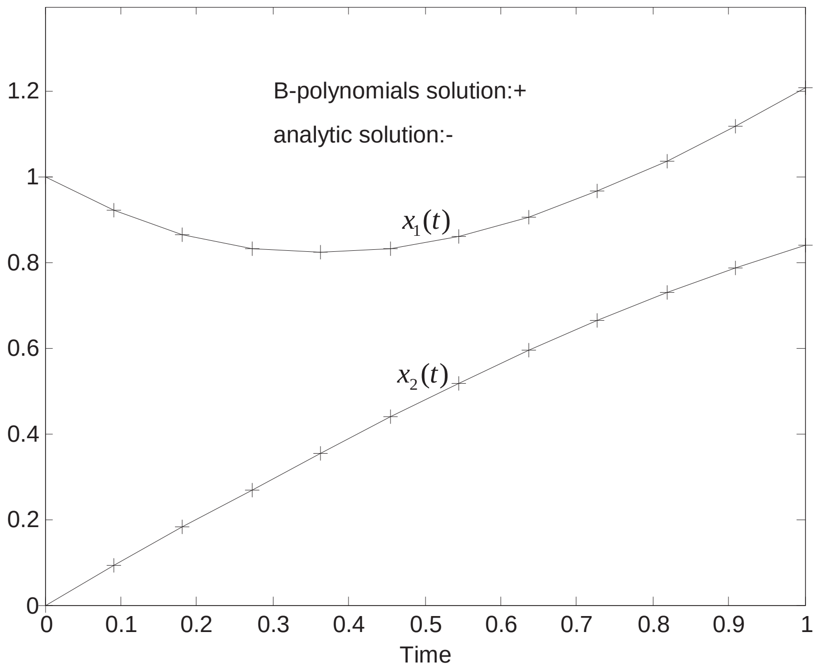

Figure 2. Bernstein polynomials and analytic solutions of index-2 descriptor system

Example 2. Consider a index-3 descriptor system (Ayaz (2004), Celik et al. (2003)) as follows:

$$\begin{bmatrix}1 & -t & t^2\\ 0 & 1 & -t\\ 0 & 0 & 0\end{bmatrix}\dot{\boldsymbol{x}}(t) = \begin{bmatrix}-1 & 1+t & -2t-t^2\\ 0 & 1 & 1-t\\ 0 & 0 & -1\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}0\\ 0\\ \sin(t)\end{bmatrix},\quad \boldsymbol{x}(0)=\begin{bmatrix}1\\1\\0\end{bmatrix}. \tag{41}$$

If we solve (41) for $\boldsymbol{x}(t)$ directly, the analytic solution for $\boldsymbol{x}(t)$ can be shown to be $\boldsymbol{x}(t) = [\mathrm{e}^{-t} + t\mathrm{e}^{t} \;\; \mathrm{e}^{t} + t\sin(t) \;\; \sin(t)]^T$. The comparison between the Bernstein polynomials solution $\tilde{\boldsymbol{x}}(t)$ and the analytic solution $\boldsymbol{x}(t)$ for $t \in [0,1]$ is shown in Figure 3 for $m = 11$, which confirms that the Bernstein polynomials gives almost the same solution of the analytic method. The average relative errors of our method and Ayaz's method (Ayaz (2004)) are $[5.538804234150940 \cdot 10^{-9} \;\; 1.291719979797174 \cdot 10^{-9} \;\; 5.529381196036456 \cdot$

$10^{-11}]$ and $[1.648469274195628 \cdot 10^{-9}\ 1.083592072202599 \cdot 10^{-9}\ 1.366030673199113 \cdot 10^{-9}]$, respectively.

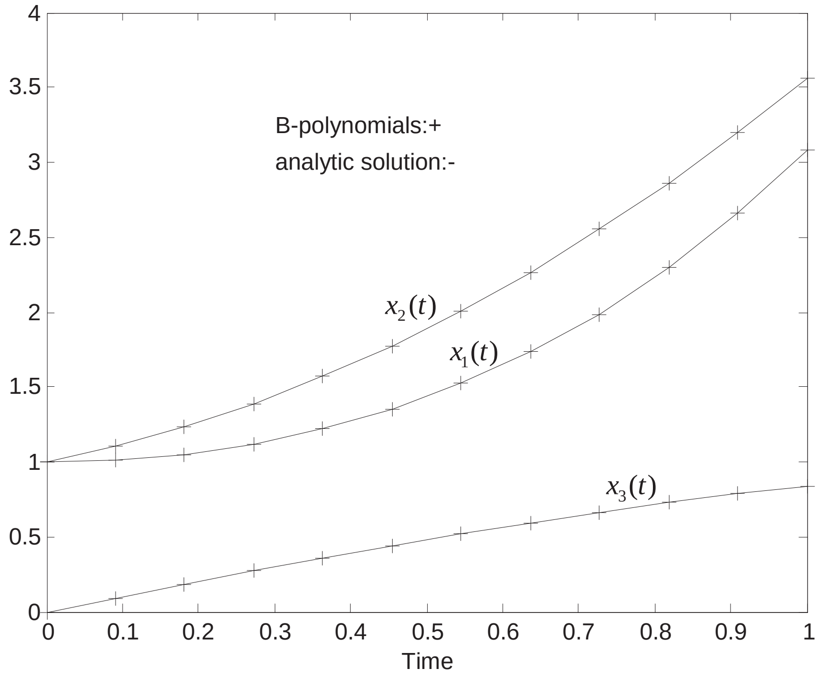

Figure 3. Bernstein polynomials and analytic solutions of index-3 descriptor system

V. Conclusion

In applying the precise properties of Bernstein polynomials, such as equations (13), (15), and (16), linear time-varying descriptor systems can be solved conveniently and accurately by using the optimal values of m systematically. The key idea is to transform the time-varying functions and its product with the states into Bernstein polynomials via product approach. It is believed that the introduction of operational matrix of integration (13), high order product matrix (15) and coefficient matrix (16) of Bernstein polynomials enlarges the application region of Bernstein polynomials quite well. Two examples have proven that Bernstein polynomials method is superior to differential transform method (Ayaz (2004)).

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